

Nonsignaling as the consistency condition for local quasi classical probability modelling of a general multipartite correlation scenario

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April 6, 2012

Abstract

We specify for a general correlation scenario a particular type of a local quasi hidden variable (LqHV) model [*J. Math. Phys.* **53** (2012), 022201] – a deterministic LqHV model, where all joint probability distributions of a correlation scenario are simulated via a single measure space with a normalized bounded real-valued measure not necessarily positive and random variables, each depending only on a setting of the corresponding measurement at the corresponding site. We prove that an arbitrary multipartite correlation scenario admits a deterministic LqHV model if and only if all its joint probability distributions satisfy the consistency condition constituting the general nonsignaling condition formulated in [*J. Phys. A: Math. Theor.* **41** (2008), 445303]. This mathematical result specifies a new probability model that has the measure-theoretic structure resembling the structure of the classical probability model but incorporates the latter only as a particular case. The local version of this *quasi classical probability model* covers the probabilistic description of every nonsignaling correlation scenario, in particular, each correlation scenario on an multipartite quantum state.

1 Introduction

A possibility of the description of quantum measurements in terms of the classical probability model has been a point of intensive discussions ever since the seminal publications of von Neumann [1], Kolmogorov [2], Einstein, Podolsky and Rosen (EPR) [3] and Bell [4, 5].

Though, in the quantum physics literature, one can still find the misleading¹ claims on a peculiarity of "quantum probabilities" and "quantum events", the probabilistic description of every quantum measurement satisfies the Kolmogorov axioms [2] for the theory of probability.

Namely, every measurement on a quantum system represented initially by a state ρ on a complex separable Hilbert space \mathcal{H} is described by the probability space² $(\Lambda, \mathcal{F}_\Lambda, \text{tr}[\rho M(\cdot)])$, where Λ is a set of measurement outcomes, \mathcal{F}_Λ is a σ -algebra of observed events $F \subseteq \Lambda$ and $\text{tr}[\rho M(\cdot)] : \mathcal{F}_\Lambda \rightarrow [0, 1]$ is the probability measure with values $\text{tr}[\rho M(F)]$, $F \in \mathcal{F}_\Lambda$, each defining the probability that, under this quantum measurement, an outcome λ belongs to a set $F \in \mathcal{F}_\Lambda$. Here, M is a normalized ($M(\Lambda) = \mathbb{I}_{\mathcal{H}}$) measure with values $M(F)$, $F \in \mathcal{F}_\Lambda$, that

¹On the misleading character of such statements, see also Ref. [6].

²In the measure theory, this triple is called a measure space.

are positive operators on \mathcal{H} – that is, a normalized positive operator-valued (POV) measure³ on $(\Lambda, \mathcal{F}_\Lambda)$.

The measure-theoretic structure of the Kolmogorov axioms [2] is crucial and the probabilistic description of each measurement in every application field satisfies these probability axioms.

However, the classical probability model, which is also often named⁴ after Kolmogorov in the mathematical physics literature and where system observables and states are represented by random variables and probability measures on a single measurable space $(\Omega, \mathcal{F}_\Omega)$, describes correctly randomness in the classical statistical mechanics and many other application fields, *but fails* either to reproduce noncontextually [10] the statistical properties of all quantum observables on a Hilbert space of a dimension $\dim \mathcal{H} \geq 3$ or to simulate via random variables, each depending only on a setting of the corresponding measurement at the corresponding site, the probabilistic description of a quantum correlation scenario on an arbitrary N -partite quantum state. For details and references, see section 1.4 in [11] and the introduction in [12].

The probabilistic description⁵ of an arbitrary multipartite correlation scenario cannot be also reproduced via the classical probability model.

Note that, in the quantum theory literature, the interpretation of quantum measurements in the classical probability terms is generally referred to as a hidden variable (HV) model.

In [13], we have introduced for a general correlation scenario the notion of a *local quasi hidden variable (LqHV) model*, where locality and the measure-theoretic structure inherent to a local hidden variable (LHV) model are preserved but positivity of a simulation measure is dropped. We have proved [13] that every quantum $S_1 \times \dots \times S_N$ -setting correlation scenario admits LqHV modelling and specified the state parameter determining quantitatively a possibility of an $S_1 \times \dots \times S_N$ -setting LHV description of an N -partite quantum state.

In the present article, we develop further the LqHV approach introduced in [13]. The paper is organized as follows.

In section 2, we specify for a general multipartite correlation scenario the notion of a deterministic LqHV model, where all joint probability distributions of a correlation scenario are simulated via a single measure space with a normalized bounded real-valued measure and random variables, each depending only on a setting of the corresponding measurement at the corresponding site. We show that the existence for a general correlation scenario of some LqHV model implies the existence for this scenario of a deterministic LqHV model.

In section 3, we prove that an arbitrary multipartite correlation scenario admits a deterministic LqHV model if and only if all its joint probability distributions satisfy the consistency condition constituting the general nonsignaling condition formulated by Eq. (10) in [12].

In section 4, we summarize the main mathematical results of the present article and discuss their conceptual implication.

2 A deterministic LqHV model

Consider an N -partite correlation scenario, where each n -th of $N \geq 2$ parties (players) performs $S_n \geq 1$ measurements with outcomes $\lambda_n \in \Lambda_n$ of an arbitrary type and \mathcal{F}_{Λ_n} is a

³The description of a quantum measurement via a POV measure was introduced by Davies and Lewis [7, 8]

⁴In the probability theory, the term "Kolmogorov probability model" refers to the probabilistic description of a measurement via the Kolmogorov axioms, see, for example, in Ref. [9].

⁵For the general framework on the probabilistic description of multipartite correlation scenarios, see Ref. [12].

σ -algebra of events $F_n \subseteq \Lambda_n$ observed at n -th site. We label each measurement at n -th site by a positive integer $s_n = 1, \dots, S_n$ and each of N -partite joint measurements, induced by this correlation scenario and with outcomes $(\lambda_1, \dots, \lambda_N) \in \Lambda_1 \times \dots \times \Lambda_N$ – by an N -tuple (s_1, \dots, s_N) , where n -th component refers to a measurement at n -th site.

For concreteness, we further specify an $S_1 \times \dots \times S_N$ -setting correlation scenario by the symbol \mathcal{E}_S , where $S := S_1 \times \dots \times S_N$, and denote by $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ a probability measure, defined on the direct product⁶ $(\Lambda_1 \times \dots \times \Lambda_N, \mathcal{F}_{\Lambda_1} \otimes \dots \otimes \mathcal{F}_{\Lambda_N})$ of measurable spaces $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, $n = 1, \dots, N$, and describing an N -partite joint measurement (s_1, \dots, s_N) under a scenario \mathcal{E}_S .

Remark 1 The superscript \mathcal{E}_S at notation $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ indicates that, in contrast to a correlation scenario represented by the so-called “nonsignaling boxes” [15, 16] and described by joint probability distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)} \equiv P_{(s_1, \dots, s_N)}$, $s_1 = 1, \dots, S_1, \dots, s_N = 1, \dots, S_N$, each depending only on settings of the corresponding measurements at the corresponding sites, for a general correlation scenario \mathcal{E}_S , each distribution $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ may also depend on settings of all (or some) other measurements. The latter is, for example, the case under a classical correlation scenario with “one-sided” or “two-sided” memory [17].

If, under an N -partite joint measurement (s_1, \dots, s_N) of scenario \mathcal{E}_S only outcomes of $M < N$ parties $1 \leq n_1 < \dots < n_M \leq N$ are taken into account while outcomes of all other parties are ignored, then the joint probability distribution of outcomes observed at these M sites is described by the marginal probability distribution

$$P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(\Lambda_1 \times \dots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \dots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \dots \times \Lambda_N). \quad (1)$$

In particular,

$$P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(\Lambda_1 \times \dots \times \Lambda_{n-1} \times d\lambda_n \times \Lambda_{n+1} \times \dots \times \Lambda_N) \quad (2)$$

is the probability distribution of outcomes observed at n -th site under a joint measurement (s_1, \dots, s_N) of scenario \mathcal{E}_S .

Remark 2 Throughout this paper, for a measure τ on the direct product $(\Lambda \times \dots \times \Lambda', \mathcal{F}_\Lambda \otimes \dots \otimes \mathcal{F}_{\Lambda'})$ of some measurable spaces, we often use notation $\tau(d\lambda \times \dots \times d\lambda')$ outside of an integral. This allows us to specify easily the structure of different marginals of τ .

For the probabilistic description of a general correlation scenario, consider the following simulation model introduced in [13].

Definition 1 [13] An $S_1 \times \dots \times S_N$ -setting correlation scenario \mathcal{E}_S , with joint probability distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$, $s_1 = 1, \dots, S_1, \dots, s_N = 1, \dots, S_N$, and outcomes $(\lambda_1, \dots, \lambda_N) \in \Lambda_1 \times \dots \times \Lambda_N$ of an arbitrary type admits a local quasi hidden variable (LqHV) model if all of its joint probability distributions admit the representation

$$P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(F_1 \times \dots \times F_N) = \int_{\Omega} P_1^{(s_1)}(F_1 \mid \omega) \cdot \dots \cdot P_N^{(s_N)}(F_N \mid \omega) \nu_{\mathcal{E}_S}(d\omega), \quad (3)$$

$$F_1 \in \mathcal{F}_{\Lambda_1}, \dots, F_N \in \mathcal{F}_{\Lambda_N},$$

⁶Recall [14] that the product σ -algebra $\mathcal{F}_{\Lambda_1} \otimes \dots \otimes \mathcal{F}_{\Lambda_N}$ on $\Lambda_1 \times \dots \times \Lambda_N$ is the smallest σ -algebra generated by the set of all rectangles $F_1 \times \dots \times F_N \subseteq \Lambda_1 \times \dots \times \Lambda_N$ with measurable “sides” $F_n \in \mathcal{F}_{\Lambda_n}$, $n = 1, \dots, N$.

in terms of a single measure space $(\Omega, \mathcal{F}_\Omega, \nu_{\mathcal{E}_S})$ with a normalized bounded real-valued measure $\nu_{\mathcal{E}_S}$ and conditional probability measures $P_n^{(s_n)}(\cdot | \omega) : \mathcal{F}_{\Lambda_n} \rightarrow [0, 1]$, defined $\nu_{\mathcal{E}_S}$ -a. e. (almost everywhere) on Ω and such that, for each $s_n = 1, \dots, S_n$ and every $n = 1, \dots, N$, the function $P_n^{(s_n)}(F_n | \cdot) : \Omega \rightarrow [0, 1]$ is measurable for each $F_n \in \mathcal{F}_{\Lambda_n}$.

In a triple $(\Omega, \mathcal{F}_\Omega, \nu)$ representing a measure space, Ω is a non-empty set, \mathcal{F}_Ω is a σ -algebra of subsets of Ω and ν is a measure on a measurable space $(\Omega, \mathcal{F}_\Omega)$. A real-valued measure ν is called normalized if $\nu(\Omega) = 1$ and bounded [14] if $|\nu(F)| \leq M < \infty$ for all $F \in \mathcal{F}_\Omega$. Note that each bounded real-valued measure ν admits [14] the Jordan decomposition $\nu = \nu^+ - \nu^-$ via positive measures

$$\nu^+(F) := \sup_{F' \in \mathcal{F}_\Omega, F' \subseteq F} \nu(F'), \quad \nu^-(F) := - \inf_{F' \in \mathcal{F}_\Omega, F' \subseteq F} \nu(F'), \quad \forall F \in \mathcal{F}_\Omega, \quad (4)$$

with disjoint supports.

We stress that, in an LqHV model (3), a normalized bounded real-valued measure $\nu_{\mathcal{E}_S}$ has a *simulation character* and may, in general, depend (via the subscript \mathcal{E}_S) on measurement settings at all (or some) sites, as an example, see measure (39) in [13].

The structure of each LqHV model is such that though some values of a simulation measure $\nu_{\mathcal{E}_S}$ may be negative, the integral standing in the right-hand side of representation (3) is *non-negative* for all $F_1 \in \mathcal{F}_{\Lambda_1}, \dots, F_N \in \mathcal{F}_{\Lambda_N}$.

If, for a correlation scenario \mathcal{E}_S , there exists representation (3), where a normalized bounded real-valued measure $\nu_{\mathcal{E}_S}$ is positive (hence, is a probability measure), then this scenario admits a local hidden variable (LHV) model formulated for a general case by Eq. (26) in [12].

As it is discussed in detail in [13], the concept of an LqHV model incorporates as particular cases and generalizes in one whole both types of simulation models known in the literature – an LHV model and an affine model [18]. Note that the latter model, where all distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ of a scenario \mathcal{E}_S are expressed via the affine sum of some LHV distributions, is in principle built up on the concept of an LHV model.

Introduce now the following special type of an LqHV model.

Definition 2 An LqHV model (3) is called *deterministic* if there exist $\mathcal{F}_\Omega/\mathcal{F}_{\Lambda_n}$ -measurable functions (random variables) $f_{n, s_n} : \Omega \rightarrow \Lambda_n$, such that, in representation (3), all conditional probability measures $P_n^{(s_n)}(\cdot | \omega)$, $s_n = 1, \dots, S_n$, $n = 1, \dots, N$, have the special form

$$P_n^{(s_n)}(F_n | \omega) = \chi_{f_{n, s_n}^{-1}(F_n)}(\omega), \quad \forall F_n \in \mathcal{F}_{\Lambda_n}, \quad (5)$$

$\nu_{\mathcal{E}_S}$ -a. e. on Ω .

Here, $f^{-1}(F) = \{\omega \in \Omega | f(\omega) \in F\}$ is the preimage of a set $F \in \mathcal{F}_\Lambda$ under a mapping $f : \Omega \rightarrow \Lambda$ and $\chi_D(\cdot)$ is the indicator function of a subset $D \subseteq \Omega$, that is, $\chi_D(\omega) = 1$ for $\omega \in D$ and $\chi_D(\omega) = 0$ for $\omega \notin D$.

The notion of a deterministic LqHV model generalizes the concept of a deterministic⁷ LHV model formulated for a general multipartite correlation scenario in section 4 of [12].

⁷The terms "deterministic HV model" and "stochastic HV model" were first introduced by Fine [19] for a bipartite scenario with two settings and two outcomes per site.

From (3) and (5) it follows that if an $S_1 \times \dots \times S_N$ -setting correlation scenario \mathcal{E}_S admits a deterministic LqHV model, then all its joint probability distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ admit the representation

$$\begin{aligned} P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(F_1 \times \dots \times F_N) &= \int_{\Omega} \chi_{f_{1,s_1}^{-1}(F_1)}(\omega) \cdot \dots \cdot \chi_{f_{N,s_N}^{-1}(F_N)}(\omega) \nu_{\mathcal{E}_S}(d\omega) \\ &= \nu_{\mathcal{E}_S} \left(f_{1,s_1}^{-1}(F_1) \cap \dots \cap f_{N,s_N}^{-1}(F_N) \right), \\ F_1 &\in \mathcal{F}_{\Lambda_1}, \dots, F_N \in \mathcal{F}_{\Lambda_N}, \end{aligned} \quad (6)$$

via a normalized bounded real-valued measure $\nu_{\mathcal{E}_S}$ on some measurable space $(\Omega, \mathcal{F}_{\Omega})$ and random variables $f_{n,s_n} : \Omega \rightarrow \Lambda_n$, each depending only on a setting of s_n -th measurement at n -th site.

In a deterministic LqHV model, the relation between a simulation measure $\nu_{\mathcal{E}_S}$ and random variables f_{n,s_n} , $s_n = 1, \dots, S_n$, $n = 1, \dots, N$, modelling scenario measurements is such that *the joint probabilities of scenario events are reproduced due to (6) only via non-negative values of $\nu_{\mathcal{E}_S}$* .

Representation (6), in turn, implies that, for arbitrary bounded measurable real-valued functions $\varphi_n : \Lambda_n \rightarrow \mathbb{R}$, $n = 1, \dots, N$, the product expectation

$$\langle \varphi_1(\lambda_1) \cdot \dots \cdot \varphi_N(\lambda_N) \rangle_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)} := \int_{\Lambda} \varphi_1(\lambda_1) \cdot \dots \cdot \varphi_N(\lambda_N) P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(d\lambda_1 \times \dots \times d\lambda_N) \quad (7)$$

takes the form

$$\langle \varphi_1(\lambda_1) \cdot \dots \cdot \varphi_N(\lambda_N) \rangle_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)} = \int (\varphi_1 \circ f_{1,s_1})(\omega) \cdot \dots \cdot (\varphi_N \circ f_{N,s_N})(\omega) \nu_{\mathcal{E}_S}(d\omega), \quad (8)$$

which differs from the form of the product expectations in a deterministic LHV model (see Eq. (31) in [12]) only by the fact that a normalized bounded real-valued measure $\nu_{\mathcal{E}_S}$ in (8) does not need to be positive.

Recall [12] that, for a given correlation scenario, a deterministic LHV model constitutes the version of the local classical probability model, where *only* the observed joint probability distributions are reproduced.

Therefore, a deterministic LqHV model (6) corresponds to the local *quasi classical probability model*, where, in contrast to the local classical probability model, an "underlying" probability space is replaced by a measure space $(\Omega, \mathcal{F}_{\Omega}, \nu)$ with a normalized bounded real-valued measure ν not necessarily positive and where:

- (i) observables with a value space $(\Lambda, \mathcal{F}_{\Lambda})$ are represented *only* by such random variables $f : \Omega \rightarrow \Lambda$ for which $\nu(f^{-1}(F)) \geq 0$, $\forall F \in \mathcal{F}_{\Lambda}$;
- (ii) a joint measurement of two observables f_1, f_2 , each with a value spaces $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, is possible *if and only if* $\nu(f_1^{-1}(F_1) \cap f_2^{-1}(F_2)) \geq 0$ for all $F_n \in \mathcal{F}_{\Lambda_n}$.

The following statement is proved in appendix A.

Proposition 1 *If an $S_1 \times \dots \times S_N$ -setting correlation scenario \mathcal{E}_S admits some LqHV model (3), then it also admits a deterministic LqHV model (6).*

This statement and theorem 1 of Ref. [13] imply.

Proposition 2 An $S_1 \times \dots \times S_N$ -setting correlation scenario \mathcal{E}_S admits a deterministic LqHV model (6) if and only if, on the direct product space $(\Lambda_1^{S_1} \times \dots \times \Lambda_N^{S_N}, \mathcal{F}_{\Lambda_1}^{\otimes S_1} \otimes \dots \otimes \mathcal{F}_{\Lambda_N}^{\otimes S_N})$, there exists a normalized bounded real-valued measure⁸

$$\mu_{\mathcal{E}_S} \left(d\lambda_1^{(1)} \times \dots \times d\lambda_1^{(S_1)} \times \dots \times d\lambda_N^{(1)} \times \dots \times d\lambda_N^{(S_N)} \right), \quad (9)$$

$$\lambda_n^{(s_n)} \in \Lambda_n, \quad s_n = 1, \dots, S_n, \quad n = 1, \dots, N,$$

returning all joint probability distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ of a scenario \mathcal{E}_S as the corresponding marginals.

3 The general consistency theorem

Let us now analyze, under what condition on joint probability distributions, an arbitrary multipartite correlation scenario admits a deterministic LqHV model.

Suppose that, under an $S_1 \times \dots \times S_N$ -setting correlation scenario \mathcal{E}_S , for all joint measurements $(s_1, \dots, s_N), (s'_1, \dots, s'_N)$ with $1 \leq M < N$ common settings s_{n_1}, \dots, s_{n_M} at arbitrary sites $1 \leq n_1 < \dots < n_M \leq N$, the marginal probability distributions (1) of outcomes observed at these sites coincide, that is:

$$P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(\Lambda_1 \times \dots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \dots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \dots \times \Lambda_N) \quad (10)$$

$$= P_{(s'_1, \dots, s'_N)}^{(\mathcal{E}_S)}(\Lambda_1 \times \dots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \dots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \dots \times \Lambda_N).$$

As we discuss this in section 3 of [12], for a general correlation scenario \mathcal{E}_S with a finite number of measurement settings at each site, condition (10) does not automatically imply that the coinciding marginals, standing in the left-hand side and the right-hand side of Eq. (10), depend only on settings of measurements s_{n_1}, \dots, s_{n_M} at sites $1 \leq n_1 < \dots < n_M \leq N$.

This means that condition (10) should be distinguished from the condition

$$P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(\Lambda_1 \times \dots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \dots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \dots \times \Lambda_N) \quad (11)$$

$$\equiv P_{(s_{n_1}, \dots, s_{n_M})}(d\lambda_{n_1} \times \dots \times d\lambda_{n_M}),$$

$$s_1 = 1, \dots, S_1, \dots, s_N = 1, \dots, S_N, \quad M = 1, \dots, N,$$

usually argued in the literature to follow if $M < N$ from condition (10).

Though condition (11) implies condition (10), the converse is not, in general, true, see proposition 1 in Ref. [12].

In view of their physical interpretations discussed in detail in [12], we call conditions (10), (11) as the nonsignaling condition and the EPR locality condition, respectively. Moreover, since, in the literature on quantum information⁹, specifically the joint combination of conditions (10), (11) is often called as nonsignaling, in order to exclude a possible misunderstanding, we further refer to the consistency condition (10) as *the general nonsignaling condition*.

⁸See remark 2.

⁹See, for example, Refs. [15, 16, 18] and therein.

We stress – the nonsignaling condition in the sense of Refs. [15, 16] implies the general nonsignaling condition (10), but the converse of this statement is not, in general, true.

The following theorem is proved in appendix B.

Theorem 1 *An $S_1 \times \cdots \times S_N$ -setting correlation scenario \mathcal{E}_S admits a deterministic LqHV model (6) if and only if all its joint probability distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$, $s_1 = 1, \dots, S_1, \dots, s_N = 1, \dots, S_N$, satisfy the consistency condition (10) constituting the general nonsignaling condition formulated in Ref. [12].*

Consider, in particular, an $S_1 \times \cdots \times S_N$ -setting correlation scenario performed on an N -partite quantum state ρ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and described by the joint probability distributions

$$\begin{aligned} & \text{tr}[\rho\{M_1^{(s_1)}(F_1) \otimes \cdots \otimes M_N^{(s_N)}(F_N)\}], \\ & F_n \in \mathcal{F}_n, \quad s_n = 1, \dots, S_n, \quad n = 1, \dots, N, \end{aligned} \quad (12)$$

where $M_n^{(s_n)}$ is a POV^{10} measure on $(\Lambda_n, \mathcal{F}_{\Lambda_n})$ representing on a Hilbert space \mathcal{H}_n a quantum measurement s_n at n -th site.

Since every quantum correlation scenario (12) satisfies condition (10) (as well as condition (11)), theorem 1 implies.

Corollary 1 *For every quantum state ρ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and arbitrary positive integers $S_1, \dots, S_N \geq 1$, the probabilistic description of each quantum $S_1 \times \cdots \times S_N$ -setting correlation scenario (12) admits a deterministic LqHV model.*

In view of the above proposition 1, the statement of corollary 1 agrees with the statement of theorem 2 in [13].

4 Conclusions

In the present paper, we have introduced (definition 2) the notion of a deterministic LqHV model, where all joint probability distributions of a multipartite correlation scenario are simulated via a *single measure space* $(\Omega, \mathcal{F}_\Omega, \nu)$, with a normalized bounded real-valued measure ν not necessarily positive, and random variables which are *local* in the sense that each of these random variables depends only on a setting of the corresponding measurement at the corresponding site.

We have proved (theorem 1) that a general $S_1 \times \cdots \times S_N$ -setting correlation scenario admits a deterministic LqHV model if and only if all its joint probability distributions satisfy the consistency condition (10) constituting the general nonsignaling condition formulated in Ref. [12].

This general result, in particular, implies (corollary 1) that the probabilistic description of every $S_1 \times \cdots \times S_N$ -setting correlation scenario (12) on an N -partite quantum state admits modelling in local quasi classical terms.

From the conceptual point of view, these mathematical results specify a new probability model that has the measure-theoretic structure $(\Omega, \mathcal{F}_\Omega, \nu)$ resembling the structure of the

¹⁰For this notion, see the introduction.

classical probability model but reduces to the latter iff a normalized bounded real-valued measure ν is positive. In the frame of this *quasi classical probability model*:

- (i) observables with a value space $(\Lambda, \mathcal{F}_\Lambda)$ are represented *only* by such random variables $f : \Omega \rightarrow \Lambda$ for which $\nu(f^{-1}(F)) \geq 0, \forall F \in \mathcal{F}_\Lambda$;
- (ii) a joint measurement of two observables f_1, f_2 , each with a value space $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, is possible *if and only if* $\nu(f_1^{-1}(F_1) \cap f_2^{-1}(F_2)) \geq 0$ for all $F_n \in \mathcal{F}_{\Lambda_n}, n = 1, 2$.

In the *quasi classical probability model*, the relation between a simulation measure ν and random variables modelling observables is such that probabilities of the observed events are reproduced only via positive values of a normalized bounded real-valued measure ν .

The local version of the *quasi classical probability model* covers (theorem 1) the probabilistic description of each nonsignaling multipartite correlation scenario, in particular, every multipartite correlation scenario (corollary 1) on an N -partite quantum state.

5 Appendix A

Proof of proposition 1. Let a scenario \mathcal{E}_S admit an LqHV model (3). Introduce the normalized real-valued measure

$$\begin{aligned} \mu_{\mathcal{E}_S} & \left(d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)} \right) \\ & := \int_{\Omega} \left\{ \prod_{s_n=1, \dots, S_n, n=1, \dots, N} P_n^{(s_n)}(d\lambda_n^{(s_n)} \mid \omega) \right\} \nu_{\mathcal{E}_S}(d\omega). \end{aligned} \quad (\text{A1})$$

This measure is bounded (see the proof of theorem 1 in [13]) and returns all distributions $P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}$ of scenario \mathcal{E}_S as the corresponding marginals. The latter means the factorizable representation

$$\begin{aligned} P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(F_1 \times \cdots \times F_N) &= \int \chi_{F_1}(\lambda_1^{(s_1)}) \cdots \chi_{F_N}(\lambda_N^{(s_N)}) \mu_{\mathcal{E}_S}(d\lambda_1^{(1)} \\ & \quad \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)}), \\ & \quad F_1 \in \mathcal{F}_{\Lambda_1}, \dots, F_N \in \mathcal{F}_{\Lambda_N}, \end{aligned} \quad (\text{A2})$$

for all $s_n = 1, \dots, S_n, n = 1, \dots, N$. Denote

$$\begin{aligned} \tilde{\omega} &:= (\lambda_1^{(1)}, \dots, \lambda_1^{(S_1)}, \dots, \lambda_N^{(1)}, \dots, \lambda_N^{(S_N)}), \\ \tilde{\Omega} &:= \Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, \quad \mathcal{F}_{\tilde{\Omega}} := \mathcal{F}_{\Lambda_1}^{\otimes S_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}^{\otimes S_N}, \\ \tilde{\nu}_{\mathcal{E}_S}(d\tilde{\omega}) &:= \mu_{\mathcal{E}_S}(d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)}) \end{aligned} \quad (\text{A3})$$

and introduce the $\mathcal{F}_{\tilde{\Omega}}/\mathcal{F}_{\Lambda_n}$ -measurable functions $f_{n,s_n} : \tilde{\Omega} \rightarrow \Lambda_n$, each defined by the relation $f_{n,s_n}(\tilde{\omega}) = \lambda_n^{(s_n)}$. Then

$$\chi_{F_n}(\lambda_n^{(s_n)}) \equiv \chi_{f_{n,s_n}^{-1}(F_n)}(\tilde{\omega}), \quad \forall F_n \in \mathcal{F}_{\Lambda_n}, \quad (\text{A4})$$

and, in view of (A3), (A4), representation (A2) takes the form

$$\begin{aligned} P_{(s_1, \dots, s_N)}^{(\mathcal{E}_S)}(F_1 \times \cdots \times F_N) &= \int_{\Omega} \chi_{f_{1,s_1}^{-1}(F_1)}(\tilde{\omega}) \cdots \chi_{f_{N,s_N}^{-1}(F_N)}(\tilde{\omega}) \tilde{\nu}_{\mathcal{E}_S}(d\tilde{\omega}) \\ &= \tilde{\nu}_{\mathcal{E}_S} \left(f_{1,s_1}^{-1}(F_1) \cap \cdots \cap f_{N,s_N}^{-1}(F_N) \right). \end{aligned} \quad (\text{A5})$$

This proves the statement of proposition 1.

6 Appendix B

Proof of theorem 1. If an N -partite correlation scenario \mathcal{E}_S , with a setting $S = S_1 \times \dots \times S_N$, admits a deterministic LqHV model (6), then, clearly, the consistency condition (10) is fulfilled.

Conversely, let scenario \mathcal{E}_S satisfy the consistency condition (10). Consider first a bipartite ($N = 2$) scenario \mathcal{E}_S with a setting $S = S_1 \times S_2$ and joint probability distributions $P_{(s_1, s_2)}^{(\mathcal{E}_S)}$ satisfying condition (10). Since, under condition (10), marginals $P_{(s_1, 1)}^{(\mathcal{E}_S)}(F_1 \times \Lambda_2), \dots, P_{(s_1, S_2)}^{(\mathcal{E}_S)}(F_1 \times \Lambda_2)$ at site "1" coincide for all $s_2 = 1, \dots, S_2$, for simplicity of notation, we denote these coinciding marginals as

$$P_{(s_1, 1)}^{(\mathcal{E}_S)}(F_1 \times \Lambda_2) = \dots = P_{(s_1, S_2)}^{(\mathcal{E}_S)}(F_1 \times \Lambda_2) := P_{s_1}^{(\mathcal{E}_S)}(F_1), \quad (\text{B1})$$

for all $F_1 \in \mathcal{F}_{\Lambda_1}$ and each $s_1 = 1, \dots, S_1$ at site "1". The superscript \mathcal{E}_S at notation $P_{s_1}^{(\mathcal{E}_S)}$ indicates that, for a general correlation scenario, this marginal does not need to depend only on a setting of measurement s_1 at site "1" (see remark 2).

Quite similarly,

$$P_{(1, s_2)}^{(\mathcal{E}_S)}(\Lambda_1 \times F_2) = \dots = P_{(S_1, s_2)}^{(\mathcal{E}_S)}(\Lambda_1 \times F_2) := P_{s_2}^{(\mathcal{E}_S)}(F_2), \quad (\text{B2})$$

for all $F_2 \in \mathcal{F}_{\Lambda_2}$ and each $s_2 = 1, \dots, S_2$ at site "2".

Introduce the normalized bounded real-valued bounded measure $\mu_{\mathcal{E}_S}$ on $(\Lambda_1^{S_1} \times \Lambda_2^{S_2}, \mathcal{F}_{\Lambda_1}^{\otimes S_1} \otimes \mathcal{F}_{\Lambda_2}^{\otimes S_2})$ with values

$$\begin{aligned} \mu_{\mathcal{E}_S} (F_1^{(1)} \times \dots \times F_1^{(S_1)} \times F_2^{(1)} \times \dots \times F_2^{(S_2)}) \\ := \sum_{s_1, s_2} \left\{ P_{(s_1, s_2)}^{(\mathcal{E}_S)}(F_1^{(s_1)} \times F_2^{(s_2)}) \prod_{\tilde{s}_1 \neq s_1} P_{\tilde{s}_1}^{(\mathcal{E}_S)}(F_1^{(\tilde{s}_1)}) \prod_{\tilde{s}_2 \neq s_2} P_{\tilde{s}_2}^{(\mathcal{E}_S)}(F_2^{(\tilde{s}_2)}) \right\} \\ - (S_1 S_2 - 1) \prod_{s_1} P_{s_1}^{(\mathcal{E}_S)}(F_1^{(s_1)}) \prod_{s_2} P_{s_2}^{(\mathcal{E}_S)}(F_2^{(s_2)}), \end{aligned} \quad (\text{B3})$$

for all $F_n^{(s_n)} \in \mathcal{F}_{\Lambda_n}$, $s_n = 1, \dots, S_n$, $n = 1, 2$. It is easy to verify that this measure returns all joint probability distributions $P_{(s_1, s_2)}^{(\mathcal{E}_S)}$ of a bipartite nonsignaling scenario \mathcal{E}_S as the corresponding marginals. By proposition 2, this implies that a bipartite correlation scenario satisfying condition (10) admits a deterministic LqHV model.

Let $N = 3$. Consider a tripartite correlation scenario \mathcal{E}_S with a setting $S = S_1 \times S_2 \times S$ and joint probability distributions $P_{(s_1, s_2, s_N)}^{(\mathcal{E})}$ satisfying condition (10). In addition to the one-party marginals denoted similarly to notation (B2), we denote by $P_{(s_1, s_2)}^{(\mathcal{E}_S)}$, $P_{(s_1, s_3)}^{(\mathcal{E}_S)}$, $P_{(s_2, s_3)}^{(\mathcal{E})}$ the coinciding two-party marginals at the corresponding sites, that is:

$$\begin{aligned} P_{(s_1, s_2, 1)}^{(\mathcal{E}_S)}(F_1 \times F_2 \times \Lambda_3) &= \dots = P_{(s_1, s_2, S_3)}^{(\mathcal{E}_S)}(F_1 \times F_2 \times \Lambda_3) := P_{(s_1, s_2)}^{(\mathcal{E}_S)}(F_1 \times F_2), \\ P_{(s_1, 1, s_3)}^{(\mathcal{E}_S)}(F_1 \times \Lambda_2 \times F_3) &= \dots = P_{(s_1, S_2, s_3)}^{(\mathcal{E}_S)}(F_1 \times \Lambda_2 \times F_3) := P_{(s_1, s_3)}^{(\mathcal{E}_S)}(F_1 \times F_3), \\ P_{(1, s_2, s_3)}^{(\mathcal{E}_S)}(\Lambda_1 \times F_2 \times F_3) &= \dots = P_{(S_1, s_2, s_3)}^{(\mathcal{E}_S)}(\Lambda_1 \times F_2 \times F_3) := P_{(s_2, s_3)}^{(\mathcal{E})}(F_2 \times F_3), \end{aligned} \quad (\text{B4})$$

for all $F_n \in \mathcal{F}_{\Lambda_n}$, $s_n = 1, \dots, S_n$, $n = 1, 2, 3$.

Similarly to our construction of measure (B3) for a bipartite case, introduce the normalized bounded real-valued measure $\tilde{\mu}_{\mathcal{E}_S}$ on $(\Lambda_1^{S_1} \times \Lambda_2^{S_2} \times \Lambda_3^{S_3}, \mathcal{F}_{\Lambda_1}^{\otimes S_1} \otimes \mathcal{F}_{\Lambda_2}^{\otimes S_2} \otimes \mathcal{F}_{\Lambda_3}^{\otimes S_3})$ with values

$$\tilde{\mu}_{\mathcal{E}_S}(F_1^{(1)} \times \dots \times F_1^{(S_1)} \times F_2^{(1)} \times \dots \times F_2^{(S_2)} \times F_3^{(1)} \times \dots \times F_3^{(S_3)}) \quad (\text{B5})$$

$$\begin{aligned} &:= \sum_{s_1, s_2, s_3} \left\{ P_{(s_1, s_2, s_3)}^{(\mathcal{E}_S)}(F_1^{(s_1)} \times F_2^{(s_2)} \times F_3^{(s_3)}) \prod_{\tilde{s}_1 \neq s_1} P_{\tilde{s}_1}^{(\mathcal{E}_S)}(F_1^{(\tilde{s}_1)}) \prod_{\tilde{s}_2 \neq s_2} P_{\tilde{s}_2}^{(\mathcal{E}_S)}(F_2^{(\tilde{s}_2)}) \prod_{\tilde{s}_3 \neq s_3} P_{\tilde{s}_3}^{(\mathcal{E}_S)}(F_3^{(\tilde{s}_3)}) \right\} \\ &- (S_1 - 1) \prod_{s_1} P_{s_1}^{(\mathcal{E}_S)}(F_1^{(s_1)}) \sum_{s_2, s_3} \left\{ P_{(s_2, s_3)}^{(\mathcal{E}_S)}(F_2^{(s_2)} \times F_3^{(s_3)}) \prod_{\tilde{s}_2 \neq s_2} P_{\tilde{s}_2}^{(\mathcal{E}_S)}(F_2^{(\tilde{s}_2)}) \prod_{\tilde{s}_3 \neq s_3} P_{\tilde{s}_3}^{(\mathcal{E}_S)}(F_3^{(\tilde{s}_3)}) \right\} \\ &- (S_2 - 1) \prod_{s_2} P_{s_2}^{(\mathcal{E}_S)}(F_2^{(s_2)}) \sum_{s_1, s_3} \left\{ P_{(s_1, s_3)}^{(\mathcal{E}_S)}(F_1^{(s_1)} \times F_3^{(s_3)}) \prod_{\tilde{s}_1 \neq s_1} P_{\tilde{s}_1}^{(\mathcal{E}_S)}(F_1^{(\tilde{s}_1)}) \prod_{\tilde{s}_3 \neq s_3} P_{\tilde{s}_3}^{(\mathcal{E}_S)}(F_3^{(\tilde{s}_3)}) \right\} \\ &- (S_3 - 1) \prod_{s_3} P_{s_3}^{(\mathcal{E}_S)}(F_3^{(s_3)}) \sum_{s_1, s_2} \left\{ P_{(s_1, s_2)}^{(\mathcal{E}_S)}(F_1^{(s_1)} \times F_2^{(s_2)}) \prod_{\tilde{s}_1 \neq s_1} P_{\tilde{s}_1}^{(\mathcal{E}_S)}(F_1^{(\tilde{s}_1)}) \prod_{\tilde{s}_2 \neq s_2} P_{\tilde{s}_2}^{(\mathcal{E}_S)}(F_2^{(\tilde{s}_2)}) \right\} \\ &+ (2S_1S_2S_3 - S_1S_2 - S_2S_3 - S_1S_3 + 1) \prod_{s_1} P_{s_1}^{(\mathcal{E}_S)}(F_1^{(s_1)}) \prod_{s_2} P_{s_2}^{(\mathcal{E}_S)}(F_2^{(s_2)}) \prod_{s_3} P_{s_3}^{(\mathcal{E}_S)}(F_3^{(s_3)}), \end{aligned}$$

for all sets $F_n^{(s_n)} \in \mathcal{F}_{\Lambda_n}$, $s_n = 1, \dots, S_n$, $n = 1, 2, 3$. Measure $\tilde{\mu}_{\mathcal{E}_S}$ returns all joint probability distributions $P_{(s_1, s_2, s_3)}^{(\mathcal{E}_S)}$ of a tripartite nonsignaling scenario \mathcal{E}_S as the corresponding marginals. By proposition 2, the latter implies that a correlation scenario \mathcal{E}_S satisfying condition (10) admits a deterministic LqHV model.

The obvious generalization to an arbitrary N -partite case of the measure constructions used in (B3), (B5) proves the sufficiency part of theorem 1.

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